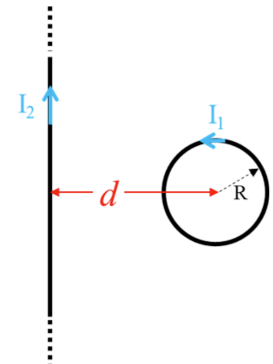
Problem Set 7

Problem 1: Magnetic Flux and Mutual Inductance

A loop of radius R and current I_1 is in the same plane and located a distance d from an infinitely long wire carrying current I_2 as shown in the figure below.



First we define the x-axis as the horizontal axis through the center of the circle, the z-axis as the vertical axis through the center of the circle and the y-axis as going into the page, where the center of the circle is *(0, 0, 0)*. We note that each point along a very thin vertical strip of dx thickness is equidistant to the wire *(r = d + x)*, so the magnetic field is the same at each of those points.

a **Find the total magnetic flux density B going through the loop due to I_2.**
We know that the magnetic flux density for an infinitely long wire is given by

$$\overrightarrow{B} = \frac{\mu I}{2\pi r}\hat{\phi} \quad \text{so}$$

$$\overrightarrow{B} = \frac{\mu_0 I_2}{2\pi(d+x)}(\hat{y})$$

b **Find the mutual inductance between the loop and the wire.**
First we find the differential magnetic flux through each very thin vertical strip of the loop. The height of half of each strip is $\sqrt{R^2 - x^2}$, so we multiply by 2. To visualize this, you can draw a triangle inside of the loop.

$$d\Phi_{12} = \overrightarrow{B} \cdot d\overrightarrow{s} = \frac{\mu_0 I_2}{2\pi(d+x)}(\hat{y}) \cdot (\hat{y}) 2\sqrt{R^2 - x^2}dx = \frac{\mu_0 I_2 \sqrt{R^2 - x^2}}{\pi(d+x)}dx$$

and plug this in to find the mutual inductance $L_{12} = \frac{1}{I_2}\int_S d\Phi_{12}$

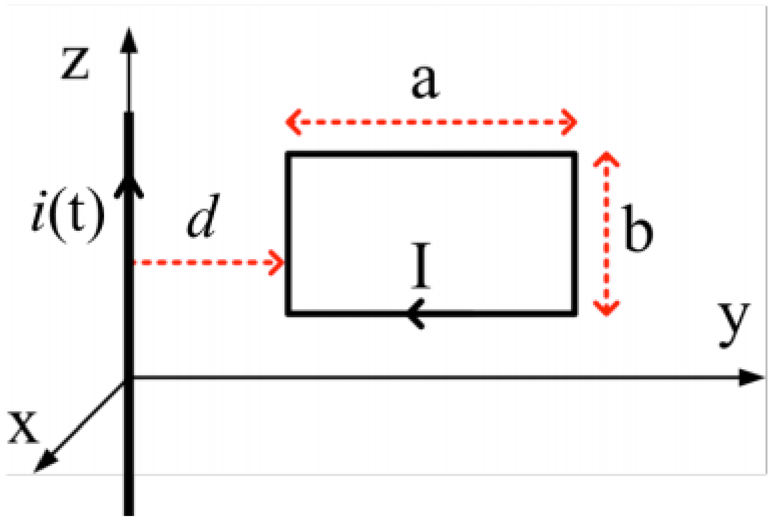
$$L_{12} = \frac{1}{I_2}\int_S \frac{\mu_0 I_2 \sqrt{R^2 - x^2}}{\pi(d+x)}dx = \frac{\mu_0}{\pi}\int_{x=-R}^{R} \frac{\sqrt{R^2 - x^2}}{(d+x)}dx$$

This is a messy integral, so using an integral table or integral calculator, we have:

$$L_{12} = \frac{\mu_0}{\pi}\left[\pi d - \pi\sqrt{d^2 - R^2}\right] = \mu_0\left[d - \sqrt{d^2 - R^2}\right]$$

Problem 2: Stationary Loop in a Time-varying Magnetic Field

An infinitely long thin wire carrying time-varying current $i(t) = I_0 \cos(\omega t)$ is located along the z-axis. An adjacent rectangualry metallic loop with dimensions (a, b) and resistivity R is located in the y-z plane at a distance d from the z-axis, as shown in the diagram below. Obtain an expression for the induced current, I, inside the loop.



We can define a new vector $\overrightarrow{r}$ that points from the z axis to an infinitesimal surface element $d\overrightarrow{s}$ within the loop.

The magnetic flux density due to the current flowing through the infinite wire (in the cylindrical coordinate) is

$$\overrightarrow{B} = \frac{\mu_0 i(t)}{2\pi r}\hat{\phi}$$

In the y-z plane where the loop resides, the magnetic flux density is going into the page which means we could re-write this equation as:

$$\overrightarrow{B} = \frac{\mu_0 i(t)}{2\pi y}(-\hat{x})$$

We can then use Faraday's law to solve for V_{emf} due to the time changing $\overrightarrow{B}$ field and invoke Ohm's law to solve for the current flowing in the loop:

$$V_{emf} = \oint_C \overrightarrow{E} \cdot d\overrightarrow{l} = -\frac{d}{dt}\int_S \overrightarrow{B} \cdot d\overrightarrow{s}$$

Where

$$I_{loop} = \frac{V_{emf}}{R}$$

And R is the resistance of the loop wire.
To evaluate $\overrightarrow{B} \cdot d\overrightarrow{s}$ inside the integral, let's first get an expression for $d\overrightarrow{s}$:

$$d\overrightarrow{s} = ds\hat{n} = -\hat{x}dydz$$

where $\hat{n}$ is in the normal direction to the surface $(-\hat{x})$ and dy times dz is our differential surface area.

$$\overrightarrow{B} \cdot d\overrightarrow{s} = \frac{\mu_0 i(t)}{2\pi y}(-\hat{x}) \cdot (-\hat{x}dydz) = \frac{\mu_0 i(t)}{2\pi y}dydz$$

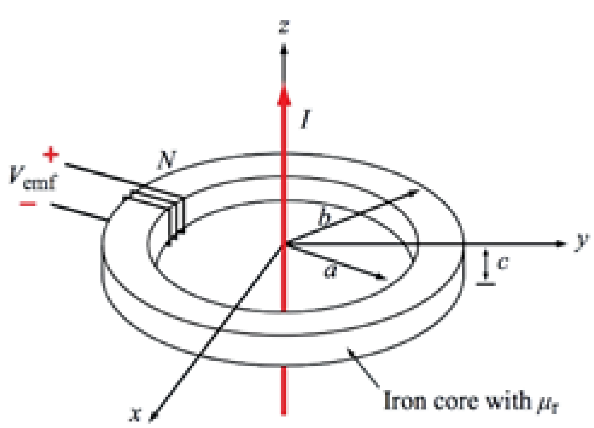
Now, we can calculate the integral over the surface enclosed by the loop:

$$\begin{aligned}\int\int_S \overrightarrow{B} \cdot d\overrightarrow{s} &= \frac{\mu_0 i(t)}{2\pi}\int_{z=z_1}^{z=z_1+b} dz \int_{y=d}^{y=d+a} \frac{dy}{y} \\ &= \frac{\mu_0 i(t)}{2\pi} b \ln\left(\frac{a+d}{d}\right)\end{aligned}$$

$$\begin{aligned}I(t) &= -\frac{1}{R}\frac{d}{dt}\left[\frac{\mu_0 i(t)}{2\pi} b \ln\left(1+\frac{a}{d}\right)\right] \\ &= \frac{I_0\omega}{R}\sin(\omega t)\frac{\mu_0}{2\pi} b \ln\left(1+\frac{a}{d}\right)\end{aligned}$$

Problem 3: Toroidal Transformer

The transformer structure shown below consists of a long wire along the z-axis carrying a time-varying current $I(t) = I_0\sin(\omega t - \frac{\pi}{3})$, coupling magnetic energy to a toroidal core situation in the x-y plane and centered at the origin, as shown in the figure below. The toroidal core uses iron material with relative permeability μ_r, around which N turns of a tightly wound coil serves to induce a voltage V_{emf}



a **Obtain an expression for V_{emf}.**

We have an infinite wire along the z-axis, carrying the time-changing current

$$I(t) = I_0 \sin(\omega t - \frac{\pi}{3})$$

Thus our magnetic field will have the form

$$\overrightarrow{H} = H_\phi(r)\hat{\phi} = \frac{I(t)}{2\pi r}\hat{\phi} \qquad \text{Cylindrical coordinates}$$

Now, we are in a magnetic material ($\mu \neq \mu_0$) in the region of interest ($a < r < b$), so the constitutive relation gives the magnetic field density:

$$\overrightarrow{B} = \mu\overrightarrow{H} = \frac{\mu I(t)}{2\pi r}\hat{\phi} \qquad \text{Inside the core}$$

We can use Faraday's Law:

$$\oint_C \overrightarrow{E} \cdot d\overrightarrow{l} = -N\frac{d}{dt}\int\int_S \overrightarrow{B} \cdot d\overrightarrow{s}$$

Note that the directions of our closed contour C and surface S must obey the right-hand rule and the polarity of V_{emf} is by the flow of current relative to an external resistor, so our surface normal is $\hat{n} = +\hat{\phi}$

$$\begin{aligned}\int\int_S \overrightarrow{B} \cdot d\overrightarrow{s} &= \int\int_S \frac{\mu_0\mu_r I(t)}{2\pi r}\hat{\phi} \cdot \left(\hat{\phi} dr dz\right) \\ &= \frac{\mu_0\mu_r I(t)}{2\pi}\int_{r=a}^{b}\frac{dr}{r}\int_{z=-c}^{0} dz \\ &= \frac{\mu_0\mu_r I(t)}{2\pi} c\ln\left(\frac{b}{a}\right)\end{aligned}$$

Now we can substitute back into Faraday's:

$$\begin{aligned}V_{emf} &= \oint_C \overrightarrow{E} \cdot d\overrightarrow{l} = -N\frac{d}{dt}\left[\frac{\mu_0\mu_r I(t)}{2\pi} c\ln\left(\frac{b}{a}\right)\right] \\ &= -N\frac{d}{dt}\left[\frac{\mu_0\mu_r I_0 \sin(\omega t - \frac{\pi}{3})}{2\pi} c\ln\left(\frac{b}{a}\right)\right] \\ V_{emf} &= -N\frac{\mu_0\mu_r I_0 \omega \cos(\omega t - \frac{\pi}{3})}{2\pi} c\ln\left(\frac{b}{a}\right)\end{aligned}$$

b **You are tasked with designing a toroidal transformer with V_{emf} = 120 V. You have access to a supply current of $I(t) = 2\sin(1000t - \frac{\pi}{3})$ A and must choose the material to fabricate the toroidal core with. Determine this material's permeability μ as a function of the toroidal transformer's geometric parameters *a, b, c,* and *N.***

Knowing $\mu = \mu_r\mu_0$ and that $I(t)$ is of the same form in part a, we can substitute the known parameters into our answer from part a.

$$120 = -N\frac{\mu(2)(1000)\cos(1000t - \frac{\pi}{3})}{2\pi} c\ln\left(\frac{b}{a}\right)$$

Solving for the material's permeability:

$$\mu = \frac{3\pi}{25Ncos(1000t - \frac{\pi}{3})c\ln\left(\frac{b}{a}\right)}$$

Because t is not given, if we assume that V_{emf} is at its maximum value (cosine term is equal to 1), we have:

$$\mu = \frac{3\pi}{25Nc\ln\left(\frac{b}{a}\right)}$$